UNIT-II

MAGNETOSTATICS

Biot-savart law, Ampere's circuital law & applications

Magnetic flux density

Maxwell's equations

Magnetic potential(vector & scalar)

Forces due to magnetic fields & Ampere's force law

Inductance & magnetic energy

Introduction :

In previous chapters we have seen that an electrostatic field is produced by static or stationary charges. The relationship of the steady magnetic field to its sources is much more complicated. The source of steady magnetic field may be a permanent magnet, a direct current or an electric field changing with time. In this chapter we shall mainly consider the magnetic field produced by a direct current. The magnetic field produced due to time varying electric field will be discussed later. Historically, the link between the electric and magnetic field was established Oersted in 1820. Ampere and others extended the investigation of magnetic effect of electricity . There are two major laws governing the magnetostatic fields are:

Biot-Savart Law, (*Ampere's Law*)

Usually, the magnetic field intensity is represented by the vector . It is customary to represent the direction of the magnetic field intensity (or current) by a small circle with a dot or cross sign depending on whether the field (or current) is out of or into the page as shown in Fig. 1.

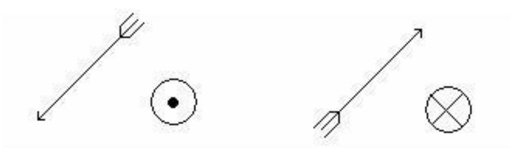


Fig. 1: Representation of magnetic field (or current)

Biot- Savart Law

This law relates the magnetic field intensity dH produced at a point due to a differential current element $Id\vec{l}$ as shown in Fig. 2.

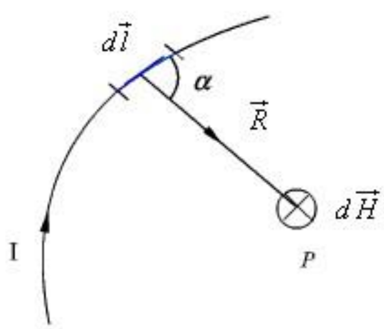


Fig. 2: Magnetic field intensity due to a current element

The magnetic field intensity $d\vec{H}$ at P can be written as,

$$d\vec{H} = \frac{Id\vec{l} \times \hat{a}_R}{4\pi R^2} = \frac{Id\vec{l} \times \vec{R}}{4\pi R^3}$$ ....(1a)

$$dH = \frac{IdlSin\alpha}{4\pi R^2}$$ ....(1b)

Where $R = |\vec{R}|$ is the distance of the current element from the point P.

Similar to different charge distributions, we can have different current distribution such as line current, surface current and volume current. These different types of current densities are shown in Fig. 3.

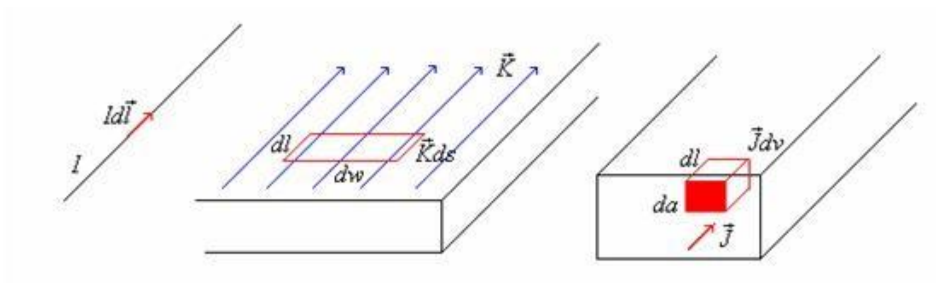


Fig. 3: Different types of current distributions

By denoting the surface current density as K (in amp/m) and volume current density as J (in amp/m2) we can write:

$Id\vec{l} = \vec{K}ds = \vec{J}dv$ ....(2)

(It may be noted that $I = Kdw = Jda$)

Employing Biot-Savart Law, we can now express the magnetic field intensity H. In terms of these current distributions.

$$\vec{H} = \int_l \frac{Id\vec{l} \times \vec{R}}{4\pi R^3}$$ .... for line current............................(3a)

$$\vec{H} = \int_S \frac{Kd\vec{s} \times \vec{R}}{4\pi R^3}$$ .... for surface current ....(3b)

$$\vec{H} = \int_V \frac{\vec{J}dv \times \vec{R}}{4\pi R^3}$$ .... for volume current......................(3c)

Ampere's Circuital Law:

Ampere's circuital law states that the line integral of the magnetic field $\vec{H}$ (circulation of H) around a closed path is the net current enclosed by this path. Mathematically,

$$\oint \vec{H}.d\vec{l} = I_{enc} \qquad \text{......................................(4)}$$

The total current I enc can be written as,

$$I_{enc} = \int_S \vec{J}.d\vec{s} \qquad \text{......................................(5)}$$

By applying Stoke's theorem, we can write

$$\oint \vec{H}.d\vec{l} = \int_S \nabla \times \vec{H}.d\vec{s}$$

$$\therefore \int_S \nabla \times \vec{H}.d\vec{s} = \int_S \vec{J}.d\vec{s} \qquad \text{......................................(6)}$$

which is the Ampere's law in the point form.

Applications of Ampere's law:

We illustrate the application of Ampere's Law with some examples.

Example : We compute magnetic field due to an infinitely long thin current carrying conductor as shown in Fig. 4. Using Ampere's Law, we consider the close path to be a circle of radius ρ as shown in the Fig. 4.

If we consider a small current element $Id\vec{l}(=Idz\hat{a}_z)$, $d\vec{H}$ is perpendicular to the plane containing both $d\vec{l}$ and $\vec{R}(=\rho\hat{a}_\rho)$. Therefore only component of $\vec{H}$ that will be present is H_ϕ ,i.e., $\vec{H} = H_\phi \hat{a}_\phi$.

By applying Ampere's law we can write,

$$\int_0^{2\pi} H_\phi \rho d\phi = H_\phi \rho 2\pi = I \qquad \text{......................................(7)}$$

Therefore, $\vec{H} = \dfrac{I}{2\pi\rho}\hat{a}_\phi$ which is same as equation (8)

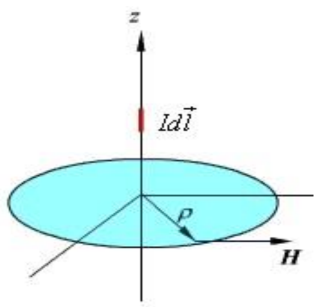


Fig. 4.:Magnetic field due to an infinite thin current carrying conductor

Example : We consider the cross section of an infinitely long coaxial conductor, the inner conductor carrying a current I and outer conductor carrying current - I as shown in figure 4.6. We compute the magnetic field as a function of ρ as follows:

In the region $0 \leq \rho \leq R_1$

$$I_{enc} = I\frac{\rho^2}{R_1^{\,2}} \quad ....(9)$$

$$H_\phi = \frac{I_{enc}}{2\pi\rho} = \frac{I\rho}{2\pi a^2} \quad ....(10)$$

In the region $R_1 \leq \rho \leq R_2$

$$I_{enc} = I$$

$$H_\phi = \frac{I}{2\pi\rho} \quad ....(11)$$

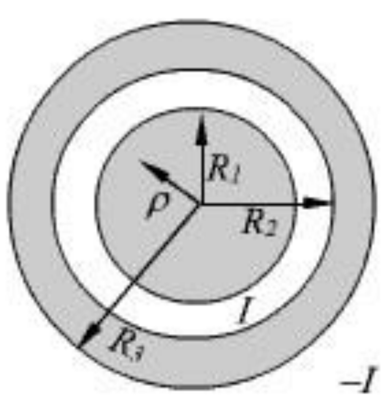


Fig. 5: Coaxial conductor carrying equal and opposite currents

In the region $R_2 \le \rho \le R_3$

$$I_{enc} = I - I\frac{\rho^2 - R_2^2}{R_3^2 - R_2^2} \quad ....(12)$$

$$H_\phi = \frac{I}{2\pi\rho}\frac{R_3^2 - \rho^2}{R_3^2 - R_2^2} \quad ....(13)$$

In the region $\rho > R_3$

$$I_{nec} = 0 \qquad H_\phi = 0 \quad ....(14)$$

Magnetic Flux Density:

In simple matter, the magnetic flux density $\vec{B}$ related to the magnetic field intensity $\vec{H}$ as $\vec{B} = \mu\vec{H}$ where μ called the permeability. In particular when we consider the free space $\vec{B} = \mu_0\vec{H}$ where $\mu_0 = 4\pi\times10^{-7}$ H/m is the permeability of the free space. Magnetic flux density is measured in terms of Wb/m 2 .

The magnetic flux density through a surface is given by:

$$\psi = \int_s \vec{B}.d\vec{s} \text{ Wb} \quad ....(15)$$

In the case of electrostatic field, we have seen that if the surface is a closed surface, the net flux passing through the surface is equal to the charge enclosed by the surface. In case of magnetic field isolated magnetic charge (i. e. pole) does not exist. Magnetic poles always occur in pair (as N-S). For example, if we desire to have an isolated magnetic pole by dividing the magnetic bar successively into two, we end up with pieces each having north (N) and south (S) pole as shown in Fig. 6 (a). This process could be continued until the magnets are of atomic dimensions; still we will have N-S pair occurring together. This means that the magnetic poles cannot be isolated.

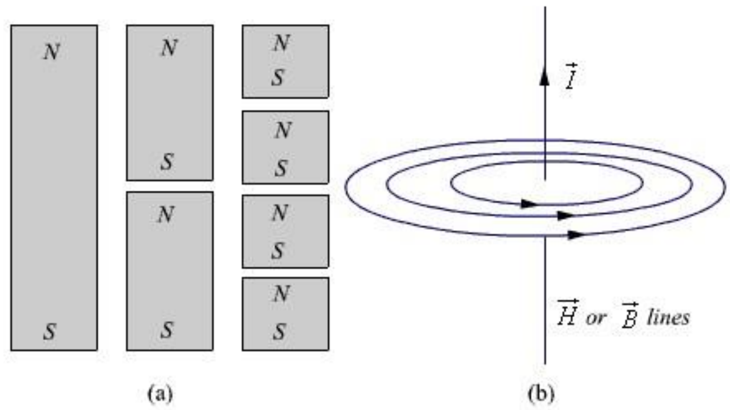


Fig. 6: (a) Subdivision of a magnet (b) Magnetic field/ flux lines of a straight current carrying conductor

Similarly if we consider the field/flux lines of a current carrying conductor as shown in Fig. 6 (b), we find that these lines are closed lines, that is, if we consider a closed surface, the number of flux lines that would leave the surface would be same as the number of flux lines that would enter the surface.

From our discussions above, it is evident that for magnetic field,

$$\oint_S \vec{B}.d\vec{s} = 0 \quad ....(16)$$

which is the Gauss's law for the magnetic field.

By applying divergence theorem, we can write:

$$\oint_S \vec{B}.d\vec{s} = \int_V \nabla.\vec{B}dv = 0$$

Hence, $\nabla.\vec{B} = 0$ ....(17)

which is the Gauss's law for the magnetic field in point form.

Magnetic Scalar and Vector Potentials:

In studying electric field problems, we introduced the concept of electric potential that simplified the computation of electric fields for certain types of problems. In the same manner let us relate the magnetic field intensity to a scalar magnetic potential and write:

$$\vec{H} = -\nabla V_m \quad ....(18)$$

From Ampere's law , we know that

$\nabla \times \vec{H} = \vec{J}$ ....(19)

Therefore, $\nabla \times (-\nabla V_m) = \vec{J}$ ....(20)

But using vector identity, $\nabla \times (\nabla V) = 0$ we find that $\vec{H} = -\nabla V_m$ is valid only where $\vec{J} = 0$. Thus the scalar magnetic potential is defined only in the region where $\vec{J} = 0$. Moreover, Vm in general is not a single valued function of position.

This point can be illustrated as follows. Let us consider the cross section of a coaxial line as shown in fig 7.

In the region $a < \rho < b$, $\vec{J} = 0$ and $\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi$

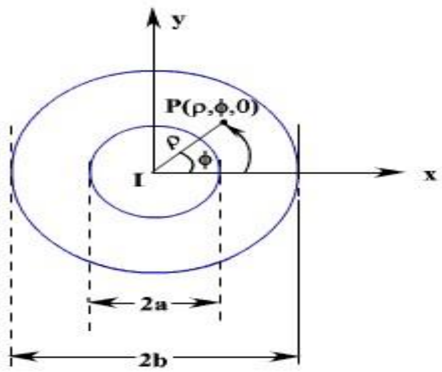


Fig. 7: Cross Section of a Coaxial Line

If Vm is the magnetic potential then,

$$-\nabla V_m = -\frac{1}{\rho}\frac{\partial V_m}{\partial \phi}$$
$$= \frac{I}{2\pi\rho}$$

If we set Vm = 0 at $\phi = 0$ then c=0 and $V_m = -\frac{I}{2\pi}\phi$

$$\therefore \text{At } \phi = \phi_0 \quad V_m = -\frac{I}{2\pi}\phi_0$$

We observe that as we make a complete lap around the current carrying conductor , we reach ϕ_0 again but Vm this time becomes

$$V_m = -\frac{I}{2\pi}(\phi_0 + 2\pi)$$

We observe that value of Vm keeps changing as we complete additional laps to pass through the same point. We introduced Vm analogous to electostatic potential V. But for static electric fields, $\nabla\times\vec{E} = 0$ and $\oint\vec{E}\cdot d\vec{l} = 0$ $\nabla\times\vec{H} = 0$, whereas for steady magnetic field $\nabla\times\vec{H} = 0$ wherever $\vec{J} = 0$ but $\oint\vec{H}\cdot d\vec{l} = I$ even if $\vec{J} = 0$ along the path of integration.

We now introduce the vector magnetic potential which can be used in regions where current density may be zero or nonzero and the same can be easily extended to time varying cases. The use of vector magnetic potential provides elegant ways of solving EM field problems.

Since $\nabla.\vec{B} = 0$ and we have the vector identity that for any vector $\vec{A}$, $\nabla.\left(\nabla\times\vec{A}\right) = 0$, we can write $\vec{B} = \nabla\times\vec{A}$.

Here, the vector field $\vec{A}$ is called the vector magnetic potential. Its SI unit is Wb/m. Thus if can find $\vec{A}$ of a given current distribution, $\vec{B}$ can be found from $\vec{A}$ through a curl operation. We have introduced the vector function $\vec{B}$ and $\vec{A}$ related its curl to $\vec{B}$. A vector function is defined fully in terms of its curl as well as divergence. The choice of $\nabla.\vec{A}$ is made as follows.

$\nabla\times\nabla\times\vec{A} = \mu\nabla\times\vec{H} = \mu\vec{J}$ ....(23)

By using vector identity, $\nabla\times\nabla\times\vec{A} = \nabla(\nabla.\vec{A}) - \nabla^2\vec{A}$ ....(24)

$\nabla(\nabla.\vec{A}) - \nabla^2\vec{A} = \mu\vec{J}$ ....(25)

Great deal of simplification can be achieved if we choose $\nabla.\vec{A} = 0$.

Putting $\nabla.\vec{A} = 0$, we get $\nabla^2\vec{A} = -\mu\vec{J}$ which is vector poisson equation.

In Cartesian coordinates, the above equation can be written in terms of the components as

$\nabla^2 A_x = -\mu J_x$ ....(26a)

$\nabla^2 A_y = -\mu J_y$ ....(26b)

$\nabla^2 A_z = -\mu J_z$ ....(26c)

The form of all the above equation is same as that of

$$\nabla^2 V = -\frac{\rho}{\varepsilon}$$ ....(27)

for which the solution is

$$V = \frac{1}{4\pi\varepsilon}\int_{V'} \frac{\rho}{R} dv', \qquad R = \left|\vec{r} - \vec{r}'\right|$$ ....(28)

In case of time varying fields we shall see that $\nabla.\vec{A} = \mu\varepsilon \frac{\partial V}{\partial t}$, which is known as Lorentz condition, V being the electric potential. Here we are dealing with static magnetic field, so $\nabla.\vec{A} = 0$.

By comparison, we can write the solution for Ax as

$$A_x = \frac{\mu}{4\pi}\int_{V'} \frac{J_x}{R} dv'$$ ....(30)

Computing similar solutions for other two components of the vector potential, the vector potential can be written as

$$\vec{A} = \frac{\mu}{4\pi}\int_{V'} \frac{\vec{J}}{R} dv'$$ ....(31)

This equation enables us to find the vector potential at a given point because of a volume current density $\vec{J}$. Similarly for line or surface current density we can write

$$\vec{A} = \frac{\mu}{4\pi}\int_{L'} \frac{I}{R} d\vec{l}'$$ ....(32)

respectively. $$\vec{A} = \frac{\mu}{4\pi}\int_{S'} \frac{\vec{K}}{R} ds'$$ ....(33)

The magnetic flux ψ through a given area S is given by

$$\psi = \int_S \vec{B}.d\vec{s}$$ ....(34)

Substituting $\vec{B} = \nabla \times \vec{A}$

$$\psi = \int_S \nabla \times \vec{A} . d\vec{s} = \oint_C \vec{A} . d\vec{l}$$ ....(35)

Vector potential thus have the physical significance that its integral around any closed path is equal to the magnetic flux passing through that path.

Inductance and Inductor:

Resistance, capacitance and inductance are the three familiar parameters from circuit theory. We have already discussed about the parameters resistance and capacitance in the earlier chapters. In this section, we discuss about the parameter inductance. Before we start our discussion, let us first introduce the concept of flux linkage. If in a coil with N closely wound turns around where a current I produces a flux ϕ and this flux links or encircles each of the N turns, the flux linkage is defined as Λ. In a linear medium $\Lambda = N\phi$, where the flux is proportional to the current, we define the self inductance L as the ratio of the total flux linkage to the current which they link.

i.e., $$L = \frac{\Lambda}{I} = \frac{N\phi}{I}$$ ....(36)

To further illustrate the concept of inductance, let us consider two closed loops C1 and C2 as shown in the figure 8, S1 and S2 are respectively the areas of C1 and C2 .

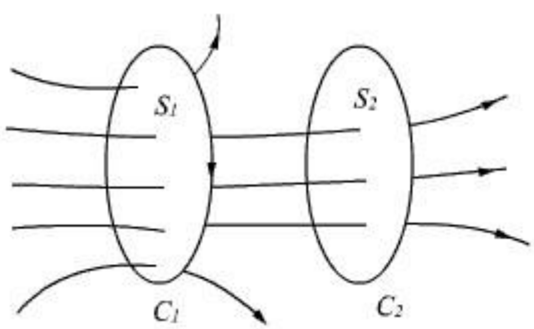


Fig:8

If a current I1 flows in C1 , the magnetic flux B1 will be created part of which will be linked to C2 as shown in Figure 8:

$$\phi_{12} = \int_{S_2} \vec{B}_1 . d\vec{S}_2$$ ....(37)

In a linear medium, ϕ_{12} is proportional to I 1. Therefore, we can write

$$\phi_{12} = L_{12} I_1$$ ....(38)

where L12 is the mutual inductance. For a more general case, if C2 has N2 turns then

$\Lambda_{12} = N_2\phi_{12}$ ....(39)

and $\Lambda_{12} = L_{12}I_1$

or $L_{12} = \dfrac{\Lambda_{12}}{I_1}$ ....(40)

i.e., the mutual inductance can be defined as the ratio of the total flux linkage of the second circuit to the current flowing in the first circuit.

As we have already stated, the magnetic flux produced in C1 gets linked to itself and if C1 has N1 turns then $\Lambda_{11} = N_1\phi_{11}$, where ϕ_{11} is the flux linkage per turn.

Therefore, self inductance

L_{11} (or L as defined earlier) $= \dfrac{\Lambda_{11}}{I_1}$ ....(41)

As some of the flux produced by I1 links only to C1 & not C2.

$\Lambda_{11} = N_1\phi_{11} > N_2\phi_{12} = \Lambda_{12}$ ....(42)

Further in general, in a linear medium, $L_{12} = \dfrac{d\Lambda_{12}}{dI_1}$ and $L_{11} = \dfrac{d\Lambda_{11}}{dI_1}$

Energy stored in Magnetic Field:

So far we have discussed the inductance in static forms. In earlier chapter we discussed the fact that work is required to be expended to assemble a group of charges and this work is stated as electric energy. In the same manner energy needs to be expended in sending currents through coils and it is stored as magnetic energy. Let us consider a scenario where we consider a coil in which the current is increased from 0 to a value I. As mentioned earlier, the self inductance of a coil in general can be written as

$L = \dfrac{d\Lambda}{di} = N\dfrac{d\phi}{di}$ ....(43a)

or $L\,di = N\,d\phi$ ....(43b)

If we consider a time varying scenario,

$$L\frac{di}{dt} = N\frac{d\phi}{dt}$$ ....(44)

We will later see that $N\frac{d\phi}{dt}$ is an induced voltage.

$\therefore v = L\frac{di}{dt}$ is the voltage drop that appears across the coil and thus voltage opposes the change of current.

Therefore in order to maintain the increase of current, the electric source must do an work against this induced voltage.

$$dW = vi\,dt$$
$$= Li\,di$$ ....(45)

$$W = \int_0^I Li\,di = \frac{1}{2}LI^2$$ (Joule)...................................(46)

which is the energy stored in the magnetic circuit.

We can also express the energy stored in the coil in term of field quantities.

For linear magnetic circuit

$$W = \frac{1}{2}\frac{N\phi}{I}I^2 = \frac{1}{2}N\phi I$$ ....(47)

Now, $$\phi = \int_s \vec{B}.d\vec{S} = BA$$ ....(48)

where A is the area of cross section of the coil. If l is the length of the coil

$$NI = Hl$$

$$\therefore W = \frac{1}{2}HBAl$$ ....(49)

Al is the volume of the coil. Therefore the magnetic energy density i.e., magnetic energy/unit volume is given by

$$W_m = \frac{W}{Al} = \frac{1}{2}BH$$ ....(50)

In vector form

$$W_m = \frac{1}{2}\vec{B}.\vec{H}$$ J/mt3 ....(51)

is the energy density in the magnetic field.

There are three ways in which the force due to magnetic fields can be experienced. The force can be

(a) Force on a charged particle:

We have F_e=QE

This shows that if Q is positive, F_e and E are in same direction. It is found that the magnetic force F_m experienced by a charge Q moving with a velocity u in magnetic field B is

F_m=Qu x B

For a moving change Q in the presence of both electric and magnetic fields, the total force on the charge is given by

F = F_e+F_m

or

F=Q(E+u x B)

This is known as Lorentz force equation.

(b) Force on a current element:

To determine the force on a current element Idl of a current carrying conductor due to the magnetic field B, we take the equation

J=P_e u

We have Idl= $\frac{dQ}{dt.}.dl = dQ = \frac{dl}{dt} = dQ.u$

Hence

Idl= dQ.u

This shows that an elemental charge dQ moving with velocity u (thereby producing convection current element dQu) is equivalent to a conduction current element Idl. Thus the force on current element is give by

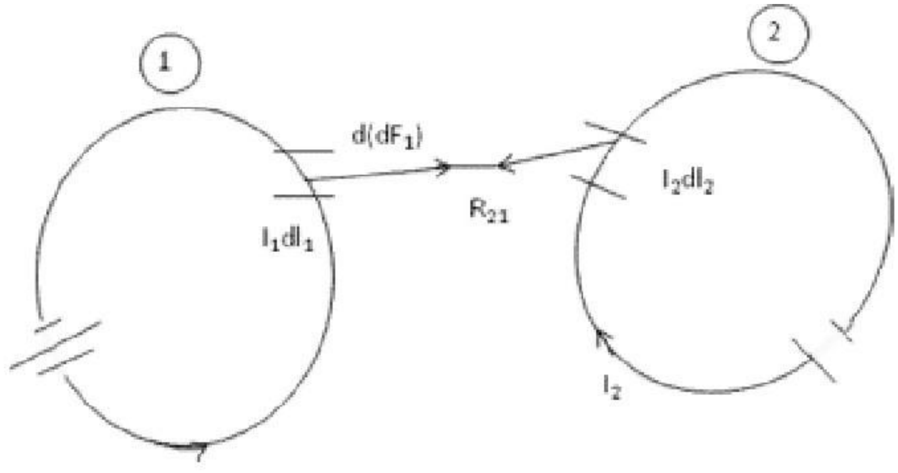
dF= Idl x B

If the current I is through a closed path L or circuit, the force on the circuit is given by

F= $\oint_L Idl \times B$

(c) Force between two current elements:

Consider the force between two elements I_1dl_1 and I_2dl_2. According to biotsavarts law, both current elements produce magnetic fields. Force $d(dF_1)$ on element I_1dl_1 due to field dB_2 produced by element I_2 dl_2 as shown in figure below:



$d(dF_1) = I_1 Dl_1 \times dB_2$
But from biot Savarts law

$$dB_2 = \frac{\mu_0 I_2 dl_2 \times a_{R21}}{4\pi R_{21}^2}$$

Hence

$$d(dF_1) = \frac{\mu_0 I_1 dl_1 \times (I_2 dl_2 \times a_{R21})}{4\pi R_{21}^2}$$

This equation is the law of force between two current elements.

We have $F1 = \frac{\mu_0 I_1 I_2 \times a_{R21}}{4\pi} \oint_{L_1}\oint_{L_2} \frac{dI_1 \times (dI_2 \times aR_{21})}{R_{21}^2}$

Inductance:

Inductance is the ability of the material to hold energy in form of magnetic field.
L, I are inductance of material and current flowing in the material.

$$E = \frac{1}{2} LI^2$$

$$\text{Inductance, L} = \frac{\text{Total flux linking current I}}{\text{current (I)}}$$

'B' is induced by I

$$\therefore \phi = \int_S \bar{B}.\bar{ds}$$

Total Flux depends on no of turns
Flux linking for n turns is 'Nϕ'.

$$\boxed{\therefore L = \frac{\lambda}{I}}$$

$\lambda = N\phi$(depending on condition i.e total Flux linking the current)

Inductance of a solenoid:

In the application of ampere's law to solenoid we found that

$$B = \frac{\mu NI}{l} Tesla$$

$$\therefore \phi = B.A = \frac{\mu NIA}{l}$$

With in a loop of N turns, the flux is linking the current N times.

$\therefore$ Total flux linking I = Nϕ

$$= \frac{\mu N^2 IA}{l}$$

$$L = \frac{\lambda}{I} = \frac{\mu N^2 A}{l}$$

Some times inductors are given for unit length as well

$$\therefore \frac{l}{l} = \mu \left(\frac{N}{l} \right)^2 .A$$

Inductance of coaxial cable:

- The total flux linking the inner and outer conductors is same as the flux in the conductor.

$$H = \frac{I}{2\pi r}(A/m)$$

$$B = \frac{\mu I}{2\pi r}(Wb/m^2)$$

Here flux density is differing with radius

$$\therefore \phi = \oint \bar{B}.d\bar{s}$$

$$\therefore \phi = \int \frac{\mu I}{2\pi r} ds$$

$$d\bar{s} = drdz\phi$$

$$\phi = \int_{z=0}^{L} \int_{r=a}^{b} \frac{\mu I}{2\pi r} drdz$$

$$\phi = \frac{\mu Il}{2\pi} \int_a^b \frac{dr}{r}$$

$$\Rightarrow \lambda = \frac{\mu Il}{2\pi} \ln\left(\frac{b}{a} \right)$$

$$\therefore L = \frac{\lambda}{I} = \frac{\mu l}{2\pi} \ln\left(\frac{b}{a} \right)$$

$$\frac{L}{l} = \frac{\mu}{2\pi} \ln\left(\frac{b}{a} \right)$$

Where μ is the permeability of medium used b/w inner and outer cores.

Also there is current flowing even inside the inner core.

$$= \frac{\mu I l \pi a^2}{8\pi} = \frac{\mu I l}{8\pi}$$

$$\therefore \frac{L_{\text{int}}}{l} = \frac{\mu}{8\pi}(H)$$

$$\frac{L_{ext}}{l} = \frac{\mu}{2\pi}\ln\left(\frac{b}{a}\right)(H/m)$$

Here μ is permeability of conductor

$$\frac{\text{Total inductance}}{Length} = \frac{L_{ext}}{l} + \frac{L_{\text{int}}}{l}$$

$$= \frac{\mu_1}{2\pi}\ln\left(\frac{\text{b}}{\text{a}}\right) + \frac{\mu_2}{2\pi}$$