Calculus of Finite Differences

2.1. DIFFERENCE SCHEMES

The difference schemes deals with the variation in the function when the independent variable changes by equal intervals. It is only a question of notation what the differences are called.

(i) Finite differences. Suppose the function $y = f(x)$ has the values $y_0, y_1, y_2, \ldots, y_n$ for the values of $x = x_0, x_0 + h, x_0 + 2h, \ldots, x_0 + nh$. To determine the values of $f(x)$ and $f'(x)$ is based on the principle of finite differences which requires following differences.

(ii) Forward differences. *The differences $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ are called first* ***forward differences*** *and are denoted by $\Delta y_0, \Delta y_1, \ldots, \Delta y_{n-1}$ where Δ is known as forward* ***difference operator***. Thus the first forward differences are given by

$$\Delta y_k = y_{k+1} - y_k.$$

Similarly, the second forward differences are

$$\Delta^2 y_k = \Delta y_{k+1} - \Delta y_k$$
$$= y_{k+2} - y_{k+1} - y_{k+1} + y_k$$
$$= y_{k+2} - 2y_{k+1} + y_k.$$

In general $$\Delta^r y_k = \Delta^{r-1} y_{k+1} - \Delta^{r-1} y_k.$$

(iii) Forward differences table :

x	y	First Diff.	Second Diff.	Third Diff.	Fourth Diff.	Fifth Diff.
x_0	y_0					
		Δy_0				
$x_0 + h$	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
$x_0 + 2h$	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
$x_0 + 3h$	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
$x_0 + 4h$	y_4		$\Delta^2 y_3$			
		Δy_4				
$x_0 + 5h$	y_5					

In this table $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ etc. are called the **leading differences**.

The operator Δ obeys the following laws :

(i) $\Delta [f(x) \pm g(x)] = \Delta f(x) \pm \Delta g(x)$

(ii) $\Delta [c f(x)] = c \Delta f(x)$, c being a constant

(iii) $\Delta(c) = 0$, c being a constant.

(iv) Backward differences. *The differences* $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ *are also called first backward differences and denoted by* $\nabla y_1, \nabla y_2, \ldots, \nabla y_n$ *respectively. Thus we have*

$$\nabla y_k = y_k - y_{k-1}$$

and

$$\nabla^2 y_k = \nabla y_k - \nabla y_{k-1}$$

In general

$$\nabla^r y_k = \nabla^{r-1} y_k - \nabla^{r-1} y_{k-1}.$$

Thus we have a backward difference table as under :

x	y	First Diff.	Second Diff.	Third Diff.	Fourth Diff.	Fifth Diff.
x_0	y_0					
		∇y_1				
$x_0 + h$	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
$x_0 + 2h$	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
$x_0 + 3h$	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
$x_0 + 4h$	y_4		$\nabla^2 y_5$			
		∇y_5				
$x_0 + 5h$	y_5					

(v) Central differences. If

$$y_1 - y_0 = \delta y_{1/2}, y_2 - y_1 = \delta y_{3/2}, \ldots, y_n - y_{n-1} = \delta y_{n-1/2}.$$

Then these differences called **central differences** and δ is called **central difference operator**.

Similarly we can define higher order central differences as

$$\delta y_{3/2} - \delta y_{1/2} = \delta^2 y_1, \delta y_{5/2} - \delta y_{3/2} = \delta^2 y_2$$

and

$$\delta^2 y_2 - \delta^2 y_1 = \delta^3 y_{3/2} \text{ and so on.}$$

The central difference table is given below :

x	y	First Diff.	Second Diff.	Third Diff.	Fourth Diff.	Fifth Diff.
x_0	y_0					
		$\delta y_{1/2}$				
x_1	y_1		$\delta^2 y_1$			
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$		
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$	
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$
x_3	y_3		$\delta^2 y_3$		$\delta^4 y_3$	
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		
x_4	y_4		$\delta^2 y_4$			
		$\delta y_{9/2}$				
x_5	y_5					

(vi) Other difference operators :

(a) Shift operator. The operator which increases the argument x by h is called shift operator

$$E\, f(x) = f(x+h)$$

and

$$E^2\, f(x) = f(x+2h)$$

$$E^3\, f(x) = f(x+3h) \ldots \text{etc.}$$

This operator E is called ***shift operator***. The inverse operator E^{-1} is defined as

$$E^{-1} f(x) = f(x-h),\ E^{-2} f(x) = f(x-2h) \ldots \text{etc.}$$

Thus in general $\quad E^n f(x) = f(x+nh)$ or $E^n y_x = y_{x+nh}$.

(b) Averaging operator. The averaging operator μ is defined as

$$\mu\, f(x) = \frac{1}{2}\left[f\left(x+\frac{1}{2}h\right)+f\left(x-\frac{1}{2}h\right)\right]$$

or

$$\mu y_x = \frac{1}{2}\,[y_{x+1/2\,h} + y_{x-1/2\,h}].$$

☞ **REMARK :**

- *The shift operator is also knows as increment operator.*

(vii) Relations between the operators. We shall have following identities :

(a) $\Delta = E - 1$ (b) $\nabla = 1 - E^{-1}$

(c) $\delta = E^{1/2} - E^{-1/2}$ (d) $\Delta = E\nabla = \nabla E = \delta E^{1/2}$

(e) $E = e^{hD}$ (f) $\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$.

Proof. (a) Since

$$\Delta y_x = y_{x+h} - y_x$$

$$= E y_x - y_x \quad \text{for all } x$$

$$= (E-1)\, y_x \quad \text{for all } x$$

$\therefore \quad \Delta = E - 1 \quad \text{or} \quad E = 1 + \Delta.$

(b) Since $\quad \nabla y_x = y_x - y_{x-h}$

$$= y_x - E^{-1} y_x \text{ for all } x$$

$$\nabla y_x = (1 - E^{-1}) y_x \text{ for all } x$$

$\therefore \quad \nabla = 1 - E^{-1}.$

(c) Since $\quad \delta y_x = y_{x+h/2} - y_{x-1/2\,h}$

$$= E^{1/2} y_x - E^{-1/2} y_x \text{ for all } x$$

$$= (E^{1/2} - E^{-1/2}) y_x \quad \text{for all } x$$

$\therefore \quad \delta = E^{1/2} - E^{-1/2}.$

(d) $\quad E\nabla y_x = E(y_x - y_{x-h})$

$$= Ey_x - Ey_{x-h} \quad \text{for all } x$$

$$= y_{x+h} - y_x \quad \text{for all } x$$

$$= \Delta y_x$$

$\therefore \quad E\nabla = \Delta \qquad \ldots(1)$

and $\quad \nabla E y_x = \nabla y_{x+h}$ for all x

$$= y_{x+h} - y_x \quad \text{for all } x$$

$$= \Delta y_x.$$

$\therefore \quad \nabla E = \Delta. \qquad \ldots(2)$

From (1) and (2)

$$\Delta = E\nabla = \nabla E.$$

(e) Since $\quad E\,f(x) = f(x+h)$

$$= f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \ldots \quad \text{[By Taylor's Theorem]}$$

$$= f(x) + h\,D\,f(x) + \frac{h^2}{2!} D^2 f(x) + \ldots$$

$$= \left(1 + hD + \frac{h^2}{2!} D^2 + \ldots\right) f(x)$$

$$E\,f(x) = e^{hD} f(x)$$

$\therefore \quad E = e^{hD}$ for all x.

(f) By the definition of averaging operator, we have

$$\mu y_x = \frac{1}{2} [y_{x+1/2\,h} + y_{x-1/2\,h}]$$

$$= \frac{1}{2} [(E^{1/2} + E^{-1/2}) y_x]$$

[By the definition of shift operator]

$\therefore \quad \mu y_x = \frac{1}{2} [E^{1/2} + E^{-1/2}]\, y_x.$

This is true for all x, therefore

$$\mu = \frac{1}{2} (E^{1/2} + E^{-1/2}).$$

(viii) The difference scheme is constructed in the following table :

x	y	First Diff.	Second Diff.	Third Diff.	Fourth Diff.	Fifth Diff.
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$			
		Δy_4				
x_5	y_5					

In the above $\Delta^k y_0$ lie on a straight line down to right. On the other hand, since $\Delta = E\nabla$, we have, $\Delta y_4 = \nabla y_5$, $\Delta^2 y_3 = \nabla^3 y_5$, $\Delta^3 y_2 = \nabla^3 y_5$ and so on, further since $\nabla^k y_n$ lie on a straight line sloping downward to the right. Similarly we also have $\Delta = E^{1/2}\,\delta$ and hence, we have $\Delta^2 y_1 = E\delta^2 y_1 = \delta^2 y_2$, $\Delta^4 y_0 = \delta^2 y_2$ and so on. In this way, we can observe that $\delta^{2k}\, y_k$ lie on a horizontal line.

(ix) Effect of an error on a difference table. Let $y_0, y_1, y_2, \ldots, y_n$ be the values of function at $x = x_0, x_1, x_2 \ldots x_n$ and ε be an error in the value y_5. Then the value of y_0 with error is $y_5 + \varepsilon$.

The table will be as under.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1 + \varepsilon$
		Δy_3		$\Delta^3 y_2 + \varepsilon$	
x_4	y_4		$\Delta^2 y_3 + \varepsilon$		$\Delta^4 y_2 - 4\varepsilon$
		$\Delta y_4 + \varepsilon$		$\Delta^3 y_3 - 3\varepsilon$	
x_5	$y_5 + \varepsilon$		$\Delta^2 y_4 - 2\varepsilon$		$\Delta^4 y_3 + 6\varepsilon$
		$\Delta y_5 - \varepsilon$		$\Delta^3 y_4 + 3\varepsilon$	
x_6	y_6		$\Delta^2 y_5 + \varepsilon$		$\Delta^4 y_4 - 4\varepsilon$
		Δy_6		$\Delta^3 y_5 + \varepsilon$	
x_7	y_7		$\Delta^2 y_6$		$\Delta^4 y_5 + \varepsilon$
		Δy_7		$\Delta^3 y_6$	
x_8	y_8		$\Delta^2 y_7$		$\Delta^4 y_6$
		Δy_8		$\Delta^2 y_7$	
x_9	y_9		$\Delta^3 y_8$		
		Δy_9			
x_{10}	y_{10}				

It is clear from above table that :

(i) The error propagates in a triangular pattern and grows rapidly.

(ii) The error increases with the increasing the order of differences.

(iii) The coefficients of ε's in any column are the binomial coefficient of $(1 - \varepsilon)^n$.

(iv) The algebraic sum of the errors in any difference column is zero.